GROUP THEORY

5TH SEMESTER - LECTURE 1
BY

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August 7, 2020

Group action

Definition

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Properties of a Group action

- $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a, \ \ \forall\, g_1, g_2 \in G, \ a \in A,$
- $e \cdot a = a, \ \ \forall a \in A.$

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- Group actions will be a powerful tool which we shall use both for proving theorems for abstract groups.
- For unravelling the structure of specific examples.
- the concept of an ”action” is a theme which will recur throughout the text as a method for studying an algebraic object by seeing how it can act on other structures.

Group action

Obsevation

Let the group G act on the set A. For each fixed $g \in G$, we get a map σ_g defined by

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- For each $g \in G$, σ_g is a **permutation** of A and $\sigma_g \in S_A$.

Proof: It is sufficient to prove that σ_g is a bijective mapping from A into A. For all $a \in A$

$$\begin{aligned}(\sigma_{g^{-1}} \circ \sigma_g) =& \sigma_{g^{-1}}(\sigma_g(a)) \\ =& g^{-1}(g \cdot a) \\ =& (g^{-1}g) \cdot a \\ =& e \cdot a = a.\end{aligned}$$

This proves $\sigma_{g^{-1}} \circ \sigma_g$ is the identity map from A to A. Since g was arbitrary, we may interchange the roles of g and g^{-1} to obtain $\sigma_g \circ \sigma_{g^{-1}}$ is also the identity map on A. Thus σ_g has a 2-sided inverse, hence is a permutation of A.

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Observation

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Proof: Define $\psi : G \to S_A$ by

$$\psi(g) = \sigma_g.$$

It is sufficient to prove that $\psi(g_1 g_2) = \psi(g_1) \circ \psi(g_2)$. From above, clearly $\sigma_g \in S_A$. Now for all $a \in A$

$$\begin{aligned}\psi(g_1 g_2)(a) =& \sigma_{g_1 g_2}(a)\\ =& (g_1 g_2) \cdot a\\ =& g_1 \cdot (g_2 \cdot a)\\ =& \sigma_{g_1}(\sigma_{g_1}(a))\\ =& (\psi(g_1) \circ \psi(g_2))(a).\end{aligned}$$

Hence, we obtain $\psi(g_1 g_2) = \psi(g_1) \circ \psi(g_2)$.

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- The homomorphism from G to S_A given above is called the **permutation representation** associated to the given action.
- This process is reversible in the sense that if $\psi : G \to S_A$ is any homomorphism from a group G to the symmetric group on a set A, then the map from $G \times A$ to A defined by

  $$g \cdot a = \psi(g)(a), \;\; \forall\, g \in G, \;\; a \in A,$$

  satisfies the properties of a group action of G on A (**check!**).

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satisfies the properties of a group action of G on A (**check!**).
- Thus actions of a group G on a set A and the homomorphisms from G into the symmetric group S_A are in bijective correspondence.
- We should also note that the definition of an action might have been more precisely named a left action since the group elements appear on the left of the set elements. We could similarly define the notion of a right action.

Examples

Let G be a group and A a nonempty set. In each of the following examples, check the properties of Group action.

1. $ga = a, \ \forall g \in G, \ a \in A$, is called the **trivial action** and G is said to act **trivially** on A. Note that distinct elements of G induce the same permutation on A. The associated permutation representation $G \to S_A$ is the trivial homomorphism which maps every element of G to the identity. If G acts on a set B and distinct elements of G induce distinct permutations of B, the action is said to be **faithful**. A **faithful action** is therefore one in which the associated permutation representation is injective.

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2. Let $V = R^n$ be a vector space over the field $F = R$, the

$$F \times V \to V$$

defined by

$$\alpha(r_1, \ldots, r_n) = (\alpha r_1, \ldots, \alpha r_n)$$

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3. $G, A = G$, define the group action $g \cdot a = ga, g \in G, a \in A$. This gives a group action of G on itself, where each $g \in G$ permutes the elements of G by left multiplication. This action is called the left regular action of G on itself. By the cancellation laws, this action is faithful.(**check!**)

Kernel

The kernel of the action of G on B is defined to be $\{g \in G | gb = b, \forall\, b \in B\}$ namely the elements of G which fix all the elements of B.

Example: For the trivial action, the kernel of the action is all of G and this action is not faithful when $|G| > 1$.

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Stabilizer

For each $a \in A$ the stabilizer of a in G is the set of elements of G that fix the element a: $\{g \in G | g \cdot a = a\}$ and is denoted by G_a.

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Group actions and permutation representations

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- In particular an action of G on A may also be viewed as a faithful action of the quotient group $G/ker\psi$ on A.
  Explanation $\psi : G \to S_A, ker\psi = \{g \in G | \sigma_g = i\}$=kernel of the group action.

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- Also the stabilizer in G of an element a of A is a subgroup of G.
- If a is a fixed element of A, then the kernel of the action is contained in the stabilizer G_a since the kernel of the action is the set of elements of G that stabilize every point, namely $\bigcap_{a \in A} G_a =$ kernel of the action.

Group action

Example

Let n be a positive integer. The group $G = S_n$ acts on the set $A = \{1, 2, \ldots, n\}$ by $\sigma \cdot i = \sigma(i)$ for all $i \in \{1, 2, \ldots, n\}$. The permutation representation associated to this action is the identity map $\psi : S_n \to S_n$, i.e., $\psi(\sigma) = \sigma$. This action is faithful because distinct element of $G = S_n$ induce distinct permutation of A. And for each $i \in \{1, 2, \ldots, n\}$ the stabilizer $G_i = \{\sigma \in S_n | \sigma(i) = i\}$ is isomorphic to S_{n-1}. (**check!**)

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Definition of a permutation representation

If G is a group, a permutation representation of G is any homomorphism of G into the symmetric group S_A for some nonempty set A. We shall say a given action of G on A affords or induces the associated permutation representation of G.

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We can think of a permutation representation as an analogue of the matrix representation of a linear transformation. In the case where A is a finite set of n elements we have $S_A \cong S_n$, so by fixing a labelling of the elements of A for instant $\{a_1, a_2, \ldots, a_n\}$, we may consider our permutations as elements of the group S_n, in the same way that fixing a basis for a vector space allows us to view a linear transformation as a matrix.

Group action

Theorem

Let G be a group acting on the nonempty set A. The relation on A defined by $a \sim b$ iff $a = g \cdot b$ for some $g \in G$ is an equivalence relation. For each $a \in A$, the number of elements in the equivalence class containing a is $[G : G_a]$, the index of the stabilizer of a.

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Proof

Clearly, we can observe that $\sim$ is an equivalence relation (check). To prove the last statement of the proposition we exhibit a bijection between the left cosets of G_a in G and the elements of the equivalence class of a. Let $C_a = \{g \cdot a | g \in G\}$. Suppose $b = g \cdot a \in C_a$. Then gG_a is a left coset of G_a in G. The map $b = g \cdot a \to gG_a$ is a map from C_a to the set of left cosets of G_a in G. This map is surjective since for any $g \in G$ the element $g \cdot a$ is an element of C_a. Since $g \cdot a = h \cdot a = (h^{-1}g)a = a$ if and only if $h^{-1}g \in G_a$ if and only if $gG_a = hG_a$ the map is also injective, hence is a bijection. This completes the proof.

Group action

By the previous Theorem a group G acting on the set A partitions A into disjoint equivalence classes under the action of G. These classes are given a name:

Definition

The equivalence class $\{g \cdot a | g \in G\}$ is called the orbit of G containing a.

Definition

The action of G on A is called transitive if there is only one orbit, i.e., given any two elements $a, b \in A$ there is some $g \in G$ such that $a = g \cdot b$.

Group action

Examples

Let G be a group acting on the set A.

1. If G acts trivially on A then $G_a = G$ for all $a \in A$ and the orbits are the elements of A. This action is transitive if and only if $|A| = 1$.

Explanation: Let $|A| > 1$ and G acts trivially on A, i.e., $g \cdot a = a$. Let $a, b \in A$ and $a \neq b$, then $g \cdot a = a$ and $g \cdot b = b$. Since the action is transitive, then $b = g \cdot a = a$ contradiction, hence $|A| = 1$.

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2. The symmetric group $G = S_n$ acts transitively in its usual action as permutations on $A = \{1, 2, \ldots, n\}$. Note that the stabilizer in G of any point i has index $n = |A|$ in S_n. (**check!**)

Explanation: $C_i = \{j | j = \sigma(i) = j\}$. If there is any other $i' \in A$, then $\tau(i) = i'$ for some $\tau \in S_n$.