SUPERPOSITION OF TWO DISTURBANCES:

Consider the propagation of two linearly polarized em waves along z axis with their elec. vectors oscillating along the x-axis,

$$\bar{E}_1 = \hat{x}\, a_1 \cos(kz - \omega t + \theta_1) \qquad -(1)$$

$$\bar{E}_2 = \hat{x}\, a_2 \cos(kz - \omega t + \theta_2) \qquad -(2)$$

The resultant is then given by, $[(\theta_1 - \theta_2) = \text{phase diff.}]$

$$\bar{E} = \bar{E}_1 + \bar{E}_2 \qquad -(3)$$

$$= \hat{x}\, a_1 \cos(kz - \omega t + \theta_1) + \hat{x}\, a_2 \cos(kz - \omega t + \theta_2)$$

$$= \hat{x}\, a \cos(kz - \omega t + \theta) \qquad -(4)$$

where $a = \left[a_1^2 + a_2^2 + 2a_1a_2\cos(\theta_1 - \theta_2)\right]^{1/2}$ $-(5)$

So the resultant (4) is also a linearly polarized wave with its elec. field vector oscillating along the same direction as its components.

Depending on the phase difference $(\theta_1 - \theta_2)$,

$a^2 = (a_1 + a_2)^2 \Rightarrow a = a_1 + a_2$ $[\theta_1 - \theta_2 = 0°]$

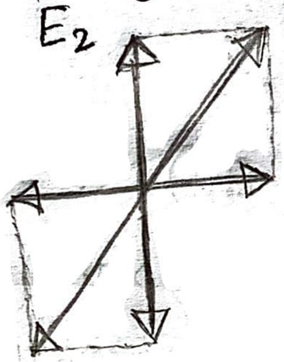
$a^2 = (a_1 \sim a_2)^2 \Rightarrow a = a_1 \sim a_2$ $[\theta_1 - \theta_2 = 180°]$

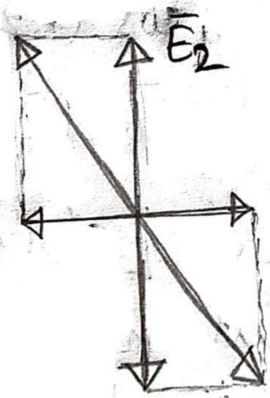
Consider two linearly polarized em waves, propagating along z-axis, but with their elec. vectors oscillating along ~~the~~ two mutually perpendicular directions,

$$\bar{E}_1 = \hat{x}\, a \cos(kz - \omega t) \qquad -(6)$$

$$\bar{E}_2 = \hat{y}\, b \cos(kz - \omega t + \theta) \qquad -(7)$$

For $\theta = m\pi$ $(m = 0, \pm 1, \pm 2, \ldots)$, the resultant will also be a linearly polarized wave with its elec. vector oscillating along a direction making a certain angle with the y-axis; this angle depending on the relative values of a & b.





To find the state of polarization of the resultant field, we consider the time variation of the resultant elec. field at any arbitrary plane perpendicular to the z-axis (say z = 0 plane)

Let E_x & E_y be the ~~resultant~~ x & y components of the resultant field $\bar{E}$ $(= \bar{E}_1 + \bar{E}_2)$, then

$$E_x = a \cos \omega t \qquad -(8)$$

$$E_y = b \cos(\omega t - \theta) \qquad -(9)$$

[z = 0 in (6) & (7).]

For $\theta = m\pi$, $E_x = a \cos \omega t$
$\& \ E_y = (-1)^m b \cos(\omega t)$ } — (10)

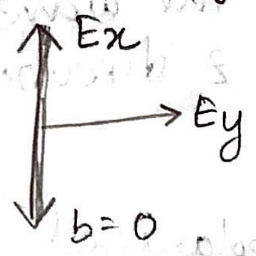
So, $\frac{E_x}{E_y} = \pm \frac{a}{b}$ (const.) — (11)

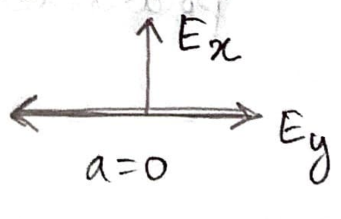
+ve sign when m is even & −ve when m is odd.

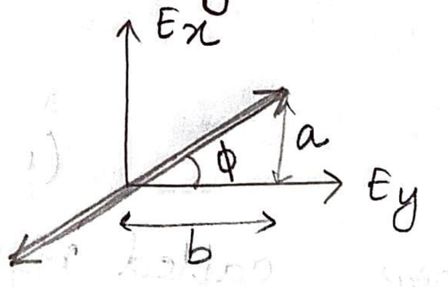
In the $E_x - E_y$ plane, (11) represents a st. line. The angle this line makes with the E_y axis is given by, $\phi = \tan^{-1}\left(\pm \frac{a}{b}\right)$ — (12)

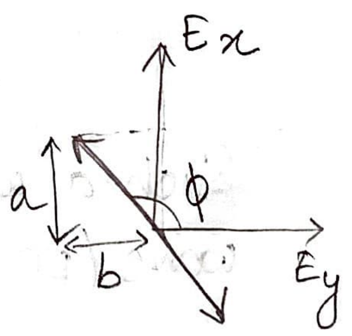
Thus, the superposition of two linearly polarized em waves with their elec. fields at right angles to each other & oscillating in phase or π out of phase, is a linearily polarized wave with its elec. vector oscillating in a direction which is, in general, diff. from the fields of either of the two waves. The tip of the resultant elec. field oscillates with angular freq. ω.

For $\theta \neq m\pi$, the resultant elec. field vector does not, in general, oscillate along a st. line.









For $\theta \neq m\pi/2$ $(m = 0, 1, 2, \ldots)$, the tip of the elec. vector rotates on the circumference of an ellipse. The ellipse would give a st. line or a circle when θ becomes an even or an odd multiple of $\pi/2$.

Case I : $\theta = \frac{\pi}{2}$ & $b = a$

$\therefore E_x = a\cos\omega t$ — (13)

$E_y = a\sin\omega t$ — (14)

$\Bigg\} E_x^2 + E_y^2 = a^2$

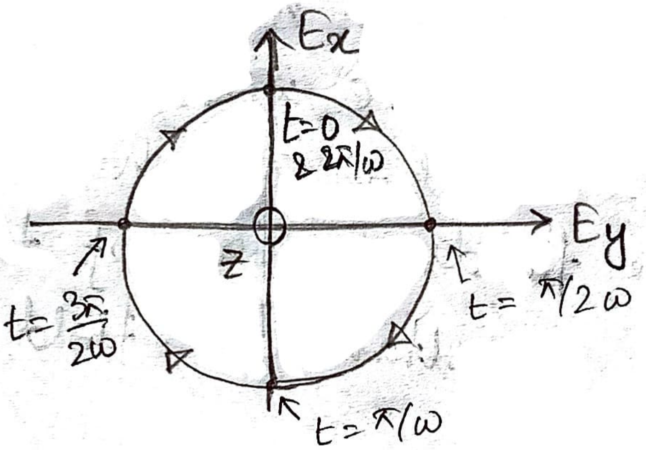
At $t=0$, $E_x = a$ & $E_y = 0$

$t = \frac{\pi}{2\omega}$, $E_x = 0$ & $E_y = a$

$t = \frac{\pi}{\omega}$, $E_x = -a$ & $E_y = 0$

$t = \frac{3\pi}{2\omega}$, $E_x = 0$ & $E_y = -a$

$t = \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \frac{6\pi}{\omega}, \ldots$ $E_x = a$ & $E_y = 0$



RCP ($\theta = \pi/2$)

Plotting the time variation of the resultant elec. vector whose x- & y- components are given by (13), (14), it can be seen that the tip of the elec. vector rotates on a circle of radius a in the clockwise direction; while the wave propagates in the +z direction (into the page.)

Such a wave is called right circularly polarized wave (RCP)

Case II : $\theta = \frac{3\pi}{2}$ & $b = a$

$\therefore E_x = a\cos\omega t$

$E_y = -a\sin\omega t$ $\Big\}$ $E_x^2 + E_y^2 = a^2$

The resultant wave in this case is a circularly polarized wave & the time variation of the resultant elec. vector shows that it is rotating in the anti-clockwise direction.

Such wave is known as Left circularly polarized (LCP) wave.

Case III : $\theta = \frac{\pi}{3}$ & $b = a$

$\therefore E_x = a\cos\omega t$

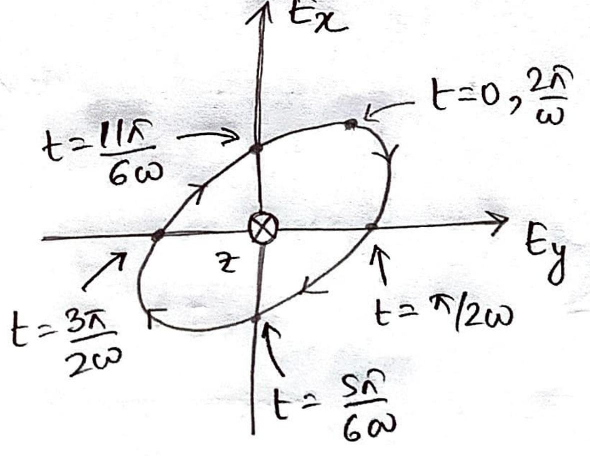
$E_y = a\cos\left(\omega t - \frac{\pi}{3}\right)$

At $t = 0$, $E_x = a$ & $E_y = \frac{1}{2}a$

$t = \frac{\pi}{3\omega}$, $E_x = \frac{1}{2}a$ & $E_y = a$

$t = \frac{\pi}{2\omega}$, $E_x = 0$ & $E_y = \frac{\sqrt{3}}{2}a$

$t = \frac{\pi}{\omega}$, $E_x = -a$ & $E_y = -\frac{1}{2}a$



REP ($\theta = \pi/3$)

The tip of the elec. field vector will rotate on the circumference of an ellipse in the clockwise direction. This is known as right elliptically polarized wave (REP).

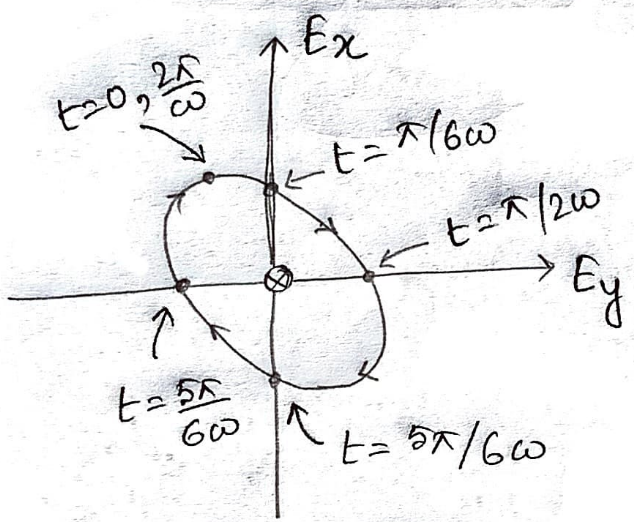
Case IV : $\theta = 2\pi/3$ & $b = a$.

$\therefore E_x = a\cos\omega t$

$E_y = a\cos\left(\omega t - \frac{2\pi}{3}\right)$

In this case, plotting E_x & E_y at various values of time, it is seen that the tip of the elec. vector will rotate on the circumference of an ellipse in the clockwise direction.

This is also a right elliptically polarized wave (REP).



REP $\left(\theta = \frac{2\pi}{3}\right)$

In elliptically polarized light, the resultant elec. field vector $\bar{E}$ will rotate, & change its magnitude, as well. The end point of $\bar{E}$ will trace out an ellipse, in a fixed plane perpendicular to the direction of propagation of the wave, $\bar{k}$.

From (8) & (9),

$$E_x = a \cos \omega t \qquad \qquad [z = 0, \text{say}]$$
$$E_y = b \cos(\omega t - \theta)$$

Then $\dfrac{E_y}{b} = \cos(\omega t - \theta) = \cos\omega t . \cos\theta - \sin\omega t . \sin\theta$

or, $\dfrac{E_y}{b} = \dfrac{E_x}{a} . \cos\theta = - \sin\omega t . \sin\theta \qquad \left[\begin{array}{l} \therefore \cos\omega t = \dfrac{E_x}{a} \\ \sin\omega t = \sqrt{1-(E_x/a)^2} \end{array} \right.$

or, $\dfrac{E_y}{b} - \dfrac{E_x}{a}\cos\theta = -\left[1 - \left(\dfrac{E_x}{a}\right)^2\right]^{1/2} \sin\theta$

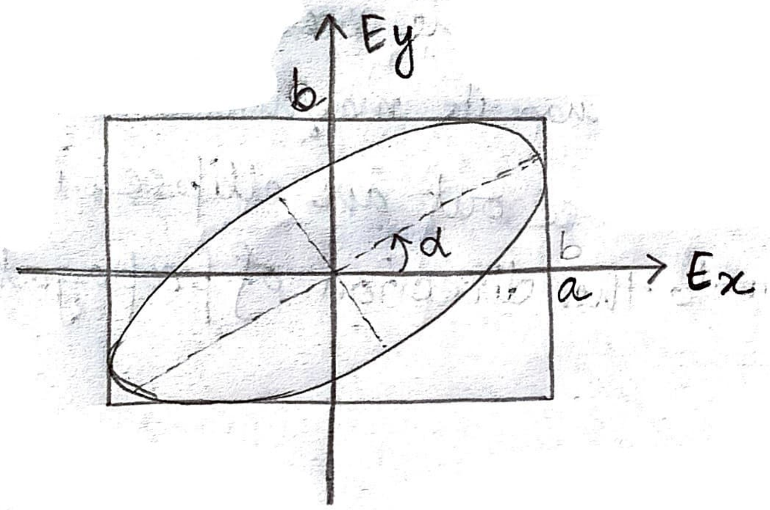
or, $\left(\dfrac{E_y}{b} - \dfrac{E_x}{a}\cos\theta\right)^2 = \left[1 - \left(\dfrac{E_x}{a}\right)^2\right] \sin^2\theta$

or, $\left(\dfrac{E_y}{b}\right)^2 + \left(\dfrac{E_x}{a}\right)^2 - \left(\dfrac{E_x}{a}\right)\left(\dfrac{E_y}{b}\right)\cos\theta = \sin^2\theta$

— (13)

The above is an eqn. of an ellipse making an angle α with the E_x axis of the (E_x, E_y)- coord. system such that,

$$\tan 2\alpha = \frac{2b.a.\cos\theta}{a^2 - b^2} \quad -(14)$$



If $\alpha = 0$ the principal axes of the ellipse aligns with the coord. axes.

For $\alpha = 0$, $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$,

from (13),

$$\frac{E_y^2}{b^2} + \frac{E_x^2}{a^2} = 1 \quad -(15)$$

For $a = b$, $E_x^2 + E_y^2 = a^2$ → eqn. of a circle. $-(16)$

If $\theta = 2m\pi$ $(m = 0, 1, 2, \ldots)$ i.e. even multiple of π,

from (13),

$$\frac{E_y}{b} - \frac{E_x}{a} = 0$$

$$\text{or, } E_y = \left(\frac{b}{a}\right)E_x \quad -(17)$$

Similarly, for odd multiples of π,

$$\frac{E_y}{b} + \frac{E_x}{a} = 0$$

or, $E_y = -\left(\frac{b}{a}\right) E_x$ — (18)

$\therefore$ (17) & (18) are eqns of st. lines having slopes $\pm\frac{b}{a}$, i.e., we have linearly polarized light as a special case of elliptically polarized light.

Linearly or plane-polarized light $\longrightarrow$ P state

Elliptically plarized $\longrightarrow$ E state.

RCP $\longrightarrow$ R state

LCP $\longrightarrow$ L state.

A linearly polarized wave can be synthesized from two oppositely polarized circular waves of equal amplitude.

A RCP wave is given by,

$$\bar{E}_1 = \hat{x}\, a\cos\omega t + \hat{y}\, a\sin\omega t \quad \text{(from (13), (14))} \quad -(19)$$

A LCP wave is given by,

$$\bar{E}_2 = \hat{x}\, a\cos\omega t - \hat{y}\, a\sin\omega t \quad -(19)$$

Adding $\bar{E}_1$ & $\bar{E}_2$,

$$\bar{E} = \bar{E}_1 + \bar{E}_2 = 2a\cos\omega t . \hat{x} \quad -(20)$$

which has a const. amplitude vector of $2a\hat{x}$ & is a P-state wave vibrating on the $\hat{x}$-axis.