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9.8. Maxima and minima.

Let I be an interval. A function $f : I \to \mathbb{R}$ is said to have a *global maximum* (or an *absolute maximum)* at a point $c \in I$ if $f(c) = \sup_{x \in I} f(x)$.

f is said to have a *global minimum* (or an *absolute minimum*) at a point $c \in I$ if $f(c) = \inf_{x \in I} f(x)$.

A function $f : I \to \mathbb{R}$ is said to have a *local maximum* (or a *relative maximum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

f is said to have a *local minimum* (or a *relative minimum*) at a point $c \in I$ if there exists a neighbourhood $N(c, \delta)$ of c such that $f(c) \leq f(x)$ for all $x \in N(c, \delta) \cap I$.

We say that f has a *local extremum* (or a *relative extremum*) at a point $c \in I$ if f has either a local maximum or a local minimum at c.

Theorem 9.8.1. Let $f : I \to \mathbb{R}$ be such that f has a local extremum at an interior point c of I. If $f'(c)$ exists then $f'(c) = 0$.

Proof. We prove the theorem for the case that f has a local maximum at c. The proof of the other case is similar.

Since $f'(c)$ exists, either $f'(c) > 0$, or $f'(c) < 0$, or $f'(c) = 0$.

Let $f'(c) > 0$. Then $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} > 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} > 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c < x < c + \delta$. Then $x - c > 0$ and therefore $f(x) > f(c)$ for all $x \in (c, c + \delta)$.

This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not> 0$ ... ... (i)

Let $f'(c) < 0$. Then $\lim_{x \to c} \dfrac{f(x) - f(c)}{x - c} < 0$.

Therefore there exists a positive δ such that $\frac{f(x)-f(c)}{x-c} < 0$ for all $x \in N'(c, \delta) \subset I$.

Let $c - \delta < x < c$. Then $x - c < 0$ and therefore $f(x) > f(c)$ for all $x \in (c - \delta, c)$.

This contradicts that f has a local maximum at c.

Consequently, $f'(c) \not< 0$ ... ... (ii)

From (i) and (ii) we have $f'(c) = 0$. This proves the theorem.

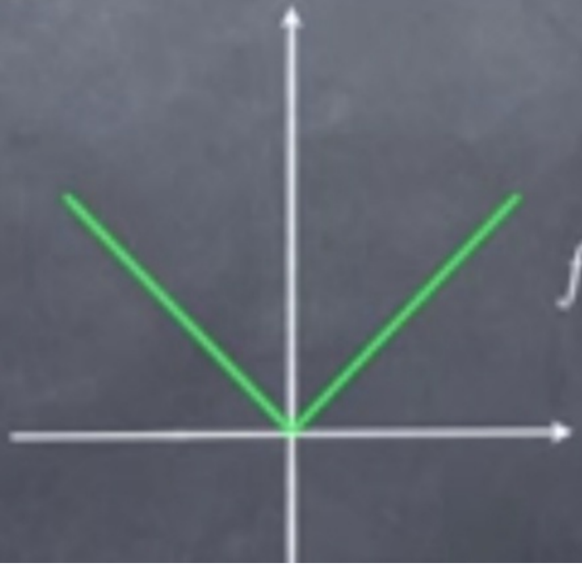
Corollary. Let $f : I \to \mathbb{R}$ and c be an interior point of I, where f has a local extremum. Then either $f'(c)$ does not exist, or $f'(c) = 0$.

Note 1. The theorem says that if the derivative $f'(c)$ exists at an interior point c of local extremum, $f'(c)$ must be 0. A function may, however have a local extremum at an interior point c of its domain without being differentiable at c. For example, the function $f(x) = | x |, x \in \mathbb{R}$ has a local minimum at 0 but $f'(0)$ does not exist.

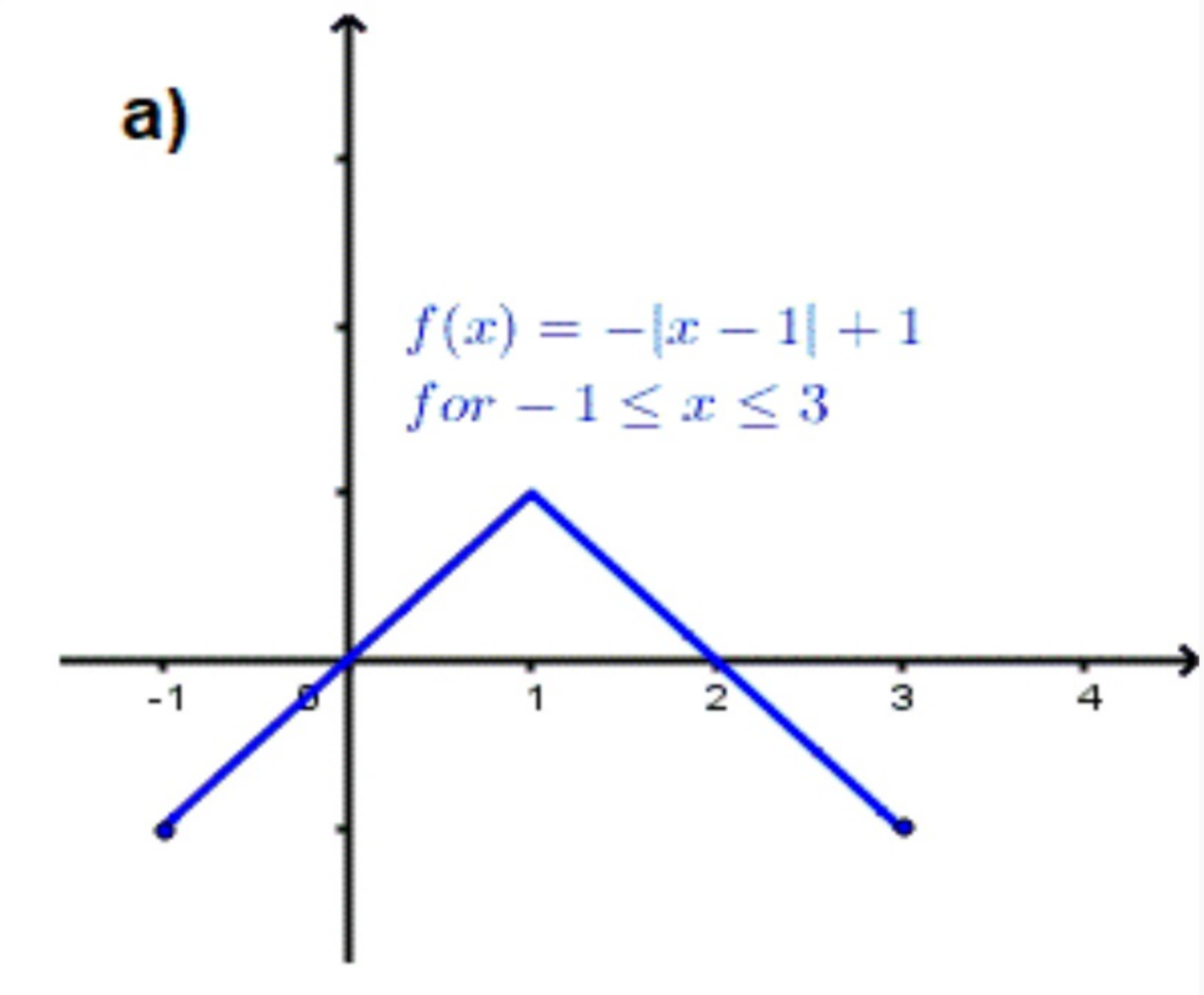
2 one sided limits $\rightsquigarrow$ 2 half-tangents (cusp)

$$\text{e.g., } f(x) = |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1 \quad \neq \quad \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

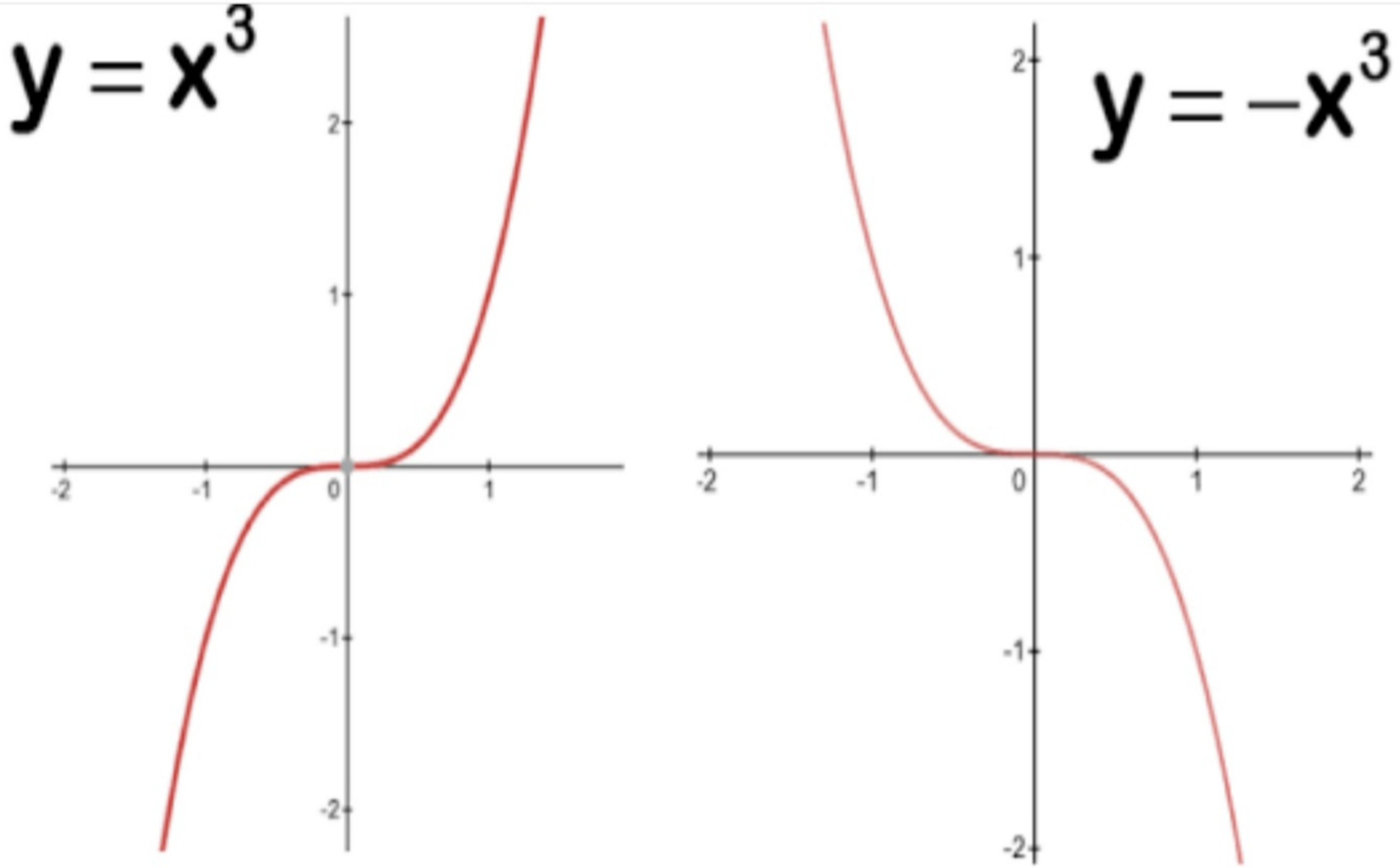


f is not differentiable at 0


a)
$f(x) = -|x-1|+1$
$for -1 \le x \le 3$
-1
0
1
2
3
4

Note 2. The condition $f'(c) = 0$ (when $f'(c)$ exists) is only a necessary condition for an interior point c to be a point of local extremum of the function f.

For example, for the function $f(x) = x^3, x \in \mathbb{R}, 0$ is an interior point of the domain of f. $f'(0) = 0$ but 0 is neither a point of local maximum nor a point of local minimum of the function f.



y = x^3
2
1
-1
-2
-2
-1
0
1
y = −x^3
2
1
-1
-2
-2
-1
0
1
2

Theorem 9.8.2. (First derivative test for extrema)

Let f be continuous on $I = [a, b]$ and c be an interior point of I. Let f be differentiable on (a, c) and (c, b).

1. If there exists a neighbourhood $(c - \delta, c + \delta) \subset I$ such that $f'(x) \geq 0$ for $x \in (c - \delta, c)$ and $f'(x) \leq 0$ for $x \in (c, c + \delta)$ then f has a local maximum at c.

2. If there exists a neighbourhood $(c - \delta, c + \delta) \subset I$ such that $f'(x) \leq 0$ for $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for $x \in (c, c + \delta)$ then f has a local minimum at c.

3. If $f'(x)$ keeps the same sign on $(c - \delta, c)$ and $(c, c + \delta)$ then f has no extremum at c.

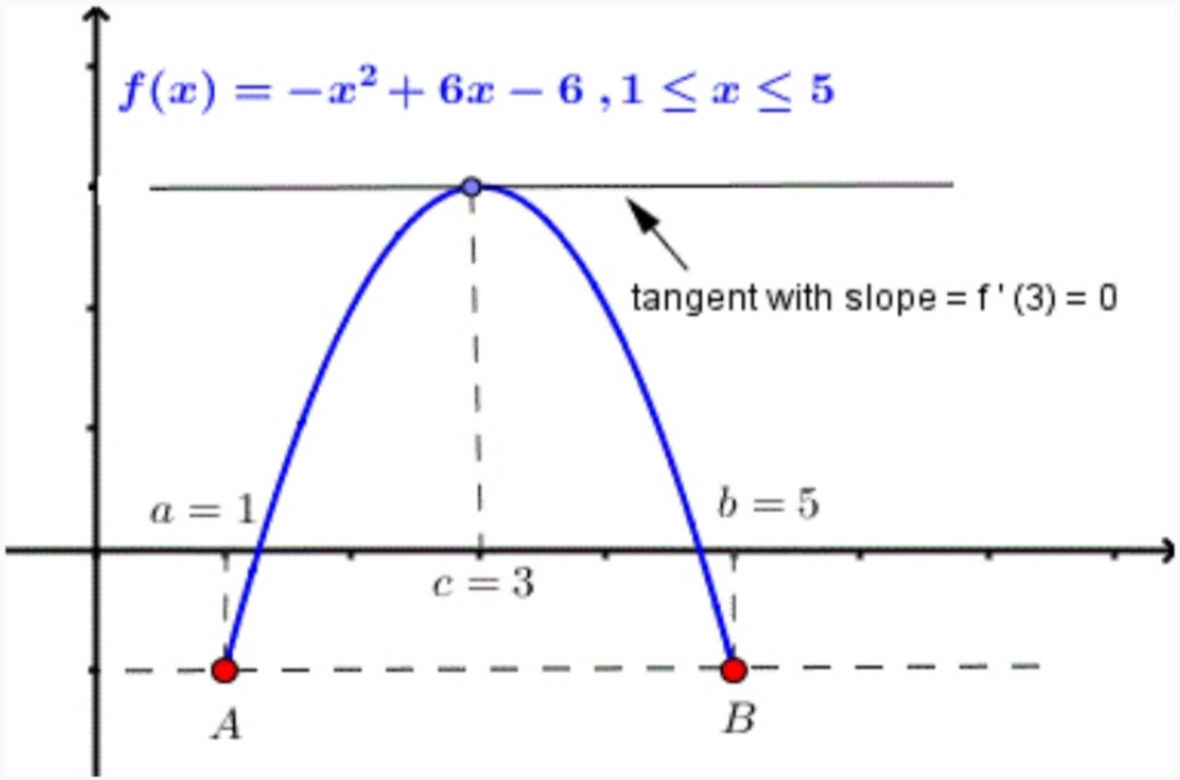


Figure 1. Rolle's theorem , example 1

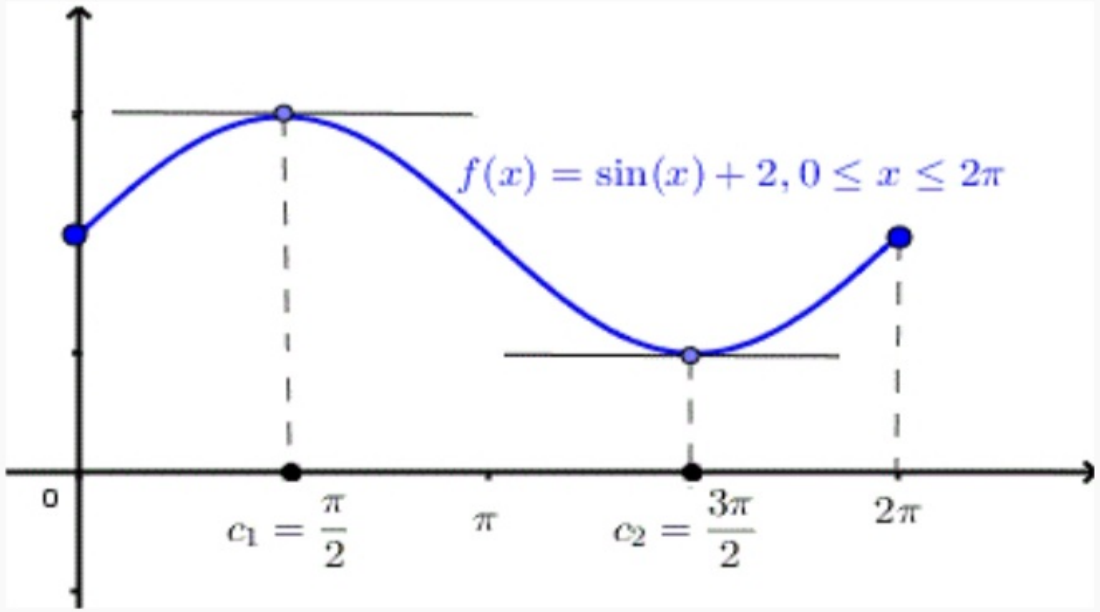


Figure 2. Rolle's theorem , example 2 with two tangents

In (Figure) we found that for $f(x) = x^2 - 2x,\ f'(x) = 2x - 2$. The graphs of these functions are shown in (Figure). Observe that $f(x)$ is decreasing for $x < 1$. For these same values of $x,\ f'(x) < 0$. For values of $x > 1,\ f(x)$ is increasing and $f'(x) > 0$. Also, $f(x)$ has a horizontal tangent at $x = 1$ and $f'(1) = 0$.

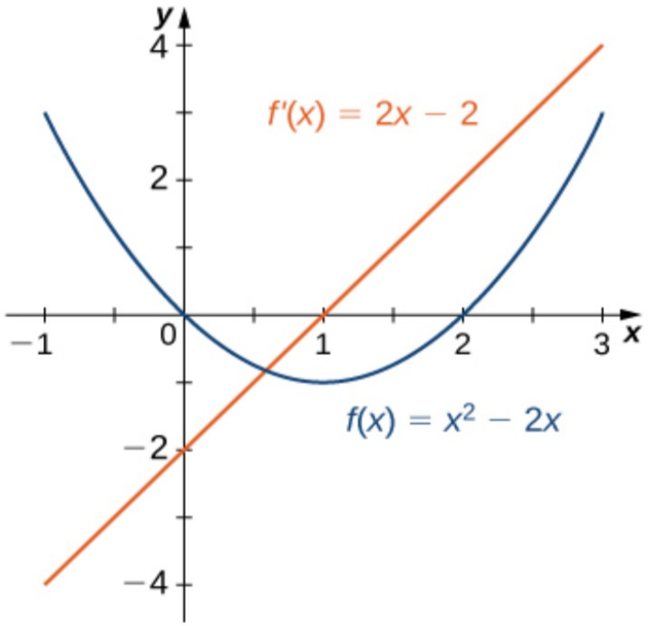


Figure 3. The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Proof. **1.** Let $x \in (c-\delta, c)$. Applying Mean value theorem to the function f on $[x, c]$, there exists a point ξ in (x, c) such that $f(c) - f(x) = (c - x)f'(\xi)$.

Since $f'(\xi) \geq 0$, we have $f(x) \leq f(c)$ for $x \in (c - \delta, c)$.

Let $x \in (c, c + \delta)$. Applying Mean value theorem to the function f on $[c, x]$, there exists a point η in (c, x) such that $f(x) - f(c) = (x - c)f'(\eta)$.

Since $f'(\eta) \leq 0$, we have $f(x) \leq f(c)$ for $x \in (c, c + \delta)$.

It follows that $f(c) \geq f(x)$ for all $x \in N(c, \delta) \cap I$.

Therefore f has a local maximum at c.

2. Similar proof.

3. Let $f'(x) > 0$ for $x \in (c-\delta, c)$ and for $x \in (c, c+\delta)$.

Then $f(x) < f(c)$ for $x \in (c-\delta, c)$ and $f(c) < f(x)$ for $x \in (c, c+\delta)$. Therefore f has neither a maximum nor a minimum at c.

Similar proof if $f'(x) < 0$ for $x \in (c-\delta, c)$ and for $(c, c+\delta)$.

This completes the proof.

Note. The converse of the theorem is not true.

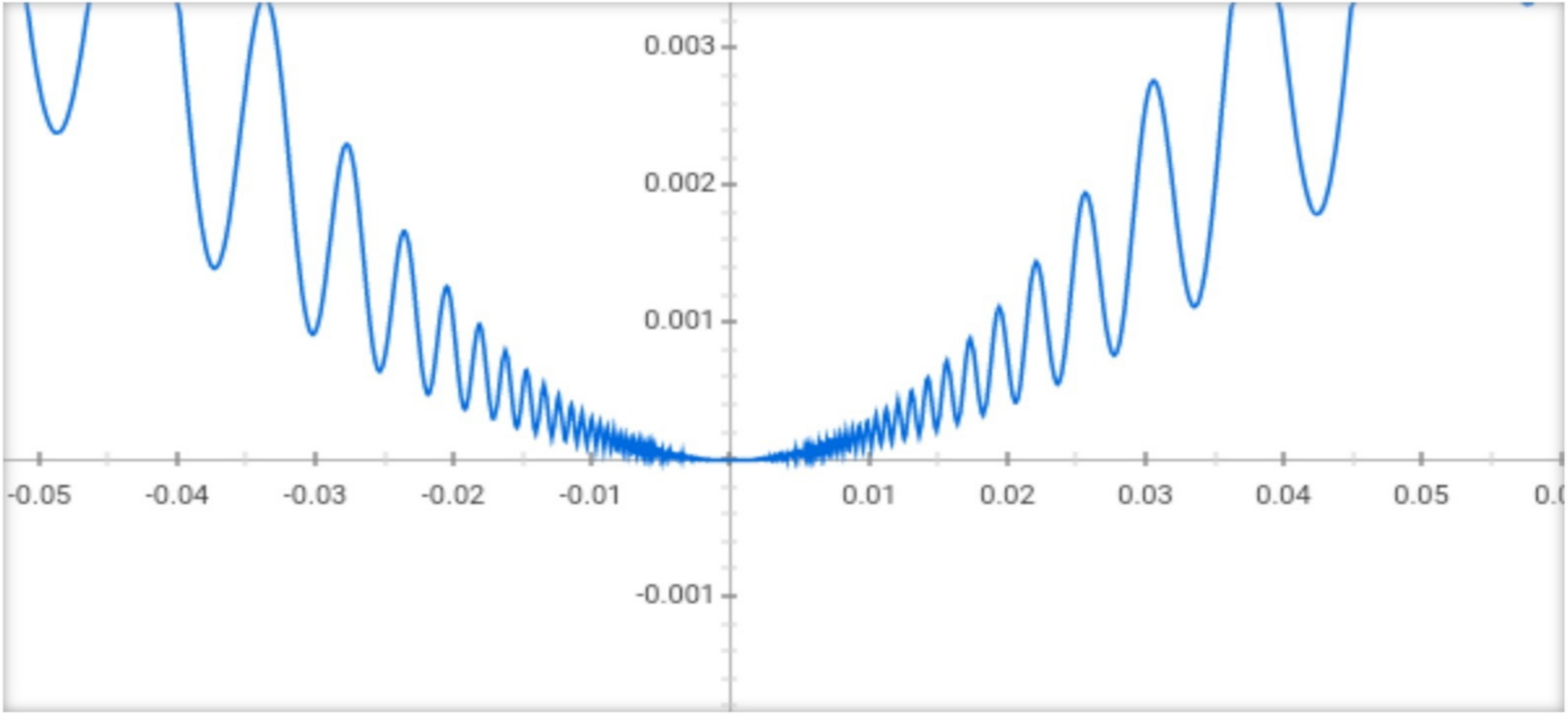
For example, let
$$\begin{aligned} f(x) &= 2x^2 + x^2 \sin \tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Then f has a local minimum at 0.

$$\begin{aligned} f'(x) &= 4x + 2x \sin \tfrac{1}{x} - \cos \tfrac{1}{x}, x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

f' takes both positive and negative values on both sides of 0 (in the immediate neighbourhood).

Graph for x^2*(2+sin(1/x))



Examples.

1. Let $f(x) = |x|, x \in \mathbb{R}$.

Then $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $x > 0$.
f is continuous at 0. Therefore f has a local minimum at 0.

2. Let $f(x) = |x - 1| + |x - 2|, x \in [0, 3]$.

f is continuous on $[0, 3]$.
$f'(x) < 0$ for $x \in (1 - \delta, 1)$, $f'(x) = 0$ for $x \in (1, 1 + \delta)$ for some δ satisfying $0 < \delta < 1$. Therefore f has a local minimum at 1.
$f'(x) = 0$ for $x \in (2 - \delta, 2)$, $f'(x) > 0$ for $x \in (2, 2 + \delta)$ for some $\delta > 0$. Therefore f has a local minimum at 2.

Note. Here f is not differentiable at 1 and 2.

3. $f(x) = (x-1)^2(x-3)^3, x \in \mathbb{R}$.

$$f'(x) = 2(x-1)(x-3)^3 + 3(x-1)^2(x-3)^2$$
$$= (x-1)(x-3)^2(5x-9), x \in \mathbb{R}.$$

f is continuous on $\mathbb{R}$. $f'(x) = 0$ at the points $1, 3, \frac{9}{5}$.

$f'(x) > 0$ for $x \in (1-\delta, 1)$ and $f'(x) < 0$ for $x \in (1, 1+\delta)$ for some $\delta > 0$.

Therefore f has a local maximum at 1.

$f'(x) > 0$ for $x \in (3-\delta, 3)$ and $f'(x) > 0$ for $x \in (3, 3+\delta)$ for some $\delta > 0$.

Therefore f has neither a maximum nor a minimum at 3.

$f'(x) < 0$ for $x \in (\frac{9}{5}-\delta, \frac{9}{5})$ and $f'(x) > 0$ for $x \in (\frac{9}{5}, \frac{9}{5}+\delta)$ for some $\delta > 0$.

Therefore f has a local minimum at $\frac{9}{5}$.

Theorem 9.8.3. (Higher order derivative test for extrema)

Let $f : I \to \mathbb{R}$ and c be an interior point of I.
If $f'(c) = f''(c) = \cdots = f^{n-1}(c) = 0$ and $f^n(c) \neq 0$, then
f has (i) no extremum at c if n be odd
(ii) a local extremum at c if n be even:
a local maximum if $f^n(c) < 0$, a local minimum if $f^n(c) > 0$.

Worked Examples.

1. $f(x) = x^5 - 5x^4 + 5x^3 + 10.$

Show that f has a maximum at 1 and a minimum at 3 and f has neither a maximum nor a minimum at 0.

For an extremum $f'(x) = 0$. $f'(x) = 0$ at $x = 1, 3, 0$.
$f''(x) = 20x^3 - 60x^2 + 30x$.
Therefore $f''(1) < 0$, $f''(3) > 0$, $f''(0) = 0$.

Since $f'(1) = 0$ and $f''(1) < 0, f$ has a local maximum at 1.
Since $f'(3) = 0$ and $f''(3) > 0, f$ has a local minimum at 3.
Since $f'(0)$ and $f''(0) = 0$, in order to decide the nature of f at 0, we are to examine derivatives of higher order at 0.
$f'''(x) = 60x^2 - 120x + 30$. $f'''(0) = 30 \neq 0$.

Therefore f has neither a maximum nor a minimum at 0.

3. Find the local extremum points of the function $f(x) = \frac{x^2}{(1-x)^3}$.

$f'(x) = \frac{2(1-x)^3x+3x^2(1-x)^2}{(1-x)^6} = \frac{x(1-x)^2(x+2)}{(1-x)^6} = \frac{x(x+2)}{(1-x)^4}$.
$f'(x) = 0$ at $x = -2, 0$.
Let h be an arbitrarily small positive number.
$f'(-2-h) > 0,\ f'(-2) = 0,\ f'(-2+h) < 0$.
$f'(0-h) < 0,\ f'(0) = 0,\ f'(0+h) > 0$.

f is continuous at -2. $f'(x) > 0$ for $x \in (-2-\delta, -2)$ and $f'(x) < 0$ for $x \in (-2, -2+\delta)$ for some $\delta > 0$.

f is continuous at 0. $f'(x) < 0$ for $x \in (-\delta, 0)$ and $f'(x) > 0$ for $x \in (0, \delta)$ for some $\delta > 0$.

Hence f has a local maximum at -2 and a local minimum at 0.

Exercises

2. Find the maximum and the minimum values of

(i) $\sin x(1+\cos x)$ in $[0, 2\pi]$ (ii) $\cos x + \cos 2x$ in $[-\frac{\pi}{4}, \frac{5\pi}{4}]$

(iii) $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x$ in $[0, \pi]$

(iv) $\cos x + \frac{1}{2}\cos 2x + \frac{1}{3}\cos 2x$ in $[0, \pi]$.

5. (i) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the base is the greatest possible?

(ii) The perimeter of an isosceles triangle is $2s$. What must its sides be so that the volume of the solid generated by revolving the triangle about the altitude upon the base is the greatest possible?

7. (i) Show that the semi-vertical angle of a right circular cone of maximum possible volume and of the given curved surface is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

(ii) Show that the semi-vertical angle of a right circular cone of minimum possible curved surface and of the given volume is $\sin^{-1}(\frac{1}{\sqrt{3}})$.

THANK YOU